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On Criticality for Competing Influences of Boundary and External Field in the Ising Model

Priscilla E. Greenwood[1] and Jiaming Sun[1]

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We continue a study of Schonmann (1994), Schonmann and Shlosman (1996), and Greenwood and Sun (1997) regarding the competing influences of boundary conditions and external field for the Ising model. We find a critical point B_0 in the competing influences for low temperature in dimension $d \geqslant 2$.

KEY WORDS: Competing influences; Ising model; Gibbs measures.

1. INTRODUCTION AND RESULTS

Consider the Ising model on Z^d. Its configuration space is $\Omega = \{-1, 1\}^{Z^d}$. Let $\mathscr{F}$ be the set of all finite subsets of Z^d. For any $\Lambda \in \mathscr{F}$, define $\Omega(\Lambda) = \{-1, 1\}^{\Lambda}$. Typical configurations in Ω or $\Omega(\Lambda)$ are denoted by $\sigma, \eta, \ldots$. Denote the value of σ at $x \in Z^d$ by σ_x. For two sites x and y in Z^d, define $|x - y| = \max\{|x_1 - y_1|, \ldots, |x_d - y_d|\}$. The energy function of the Ising model in Λ, with boundary condition η and external field s is

$$H_{\Lambda, \eta, s}(\sigma) := \frac{1}{2} \sum_{\{x, y\} \subset \Lambda, |x-y|=1} \sigma_x \sigma_y - \frac{1}{2} \sum_{x \in \Lambda;\, y \in \Lambda^c, |x-y|=1} \sigma_x \eta_y - \frac{s}{2} \sum_{x \in \Lambda} \sigma_x \tag{1.1}$$

for $\sigma \in \Omega$. Given a set $\Lambda \in \mathscr{F}$ and a configuration η, we introduce

$$\Omega_{\Lambda, \eta} := \{\sigma \in \Omega : \sigma_x = \eta_x \text{ for all } x \notin \Lambda\}$$

[1] Department of Mathematics, University of British Columbia, Vancouver, BC, V6T 1Z2 Canada; e-mail: pgreenw@math.ubc.ca; sunj@math.ubc.ca

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The Gibbs measure in Λ with boundary condition η and external field s at the temperature $T=1/\beta$ is defined on Ω as

$$\mu_{\Lambda,\eta,s}(\sigma) := \begin{cases} \exp(-\beta H_{\Lambda,\eta,s}(\sigma))/Z_{\Lambda,\eta,s} & \text{if } \sigma \in \Omega_{\Lambda,\eta} \\ 0 & \text{otherwise} \end{cases} \tag{1.2}$$

where $Z_{\Lambda,\eta,s}$ is the normalizing constant given by

$$Z_{\Lambda,\eta,s} := \sum_{\sigma \in \Omega_{\Lambda,\eta}} \exp(-\beta H_{\Lambda,\eta,s}(\sigma)) \tag{1.3}$$

Expectation with respect to $\mu_{\Lambda,\eta,s}$ is denoted by $E_{\Lambda,\eta,s}$. When $\eta \equiv -1$ or $+1$, we replace η by $-$ or $+$. It is well known that $\mu_{\Lambda,-,0}$ ($\mu_{\Lambda,+,0}$) converges weakly to the pure ($-$)-phase μ_- (pure ($+$)-phase μ_+) as Λ grows to Z^d. For $d \geqslant 2$, there is a critical temperature $T_c > 0$ such that $\mu_- = \mu_+$ if $T > T_c$ and $\mu_- \neq \mu_+$ if $T < T_c$, in which case a phase transition occurs. See Ellis (1985), Georgii (1988) or Chen (1992).

Assume that $T < T_c$ Let $\Lambda(1/h)$ be the cube in Z^d with side length $1/h$ and centered at the origin, for all $h > 0$. Consider the Gibbs measure $\mu_{\Lambda(1/h),-,s_h}$, where the external field s_h depends on h and decreases to 0 as $h \searrow 0$. What will be the possible limit of the Gibbs measure $\mu_{\Lambda(1/h),-,s_h}$ as $h \searrow 0$? Intuitively, when $h \searrow 0$, the negative boundary condition would force $\mu_{\Lambda(1/h),-,s_h}$ to converge to the pure ($-$)-phase μ_- but on the other hand the small positive external field s_h would pull $\mu_{\Lambda(1/h),-,s_h}$ to the pure ($+$)-phase μ_+. In other words, when $h \searrow 0$, the behavior of the Gibbs measure $\mu_{\Lambda(1/h),-,s_h}$ depends on the balance between the competing influences of the negative boundary condition and the positive external field s_h. This phenomenon of competing influences has been investigated by several authors. Martirosyan (1987) first proved that if one sets $s_h = Bh$, where B is a constant, then at low temperature T and with large B, the Gibbs measure $\mu_{\Lambda(1/h),-,Bh}$ converges weakly to the pure ($+$)-phase μ_+. The choice of $s_h = Bh$ is intuitively reasonable, since the negative boundary condition influences Gibbs measures by a surface order whereas the external field is of a volume order. Schonmann (1994) (referred to in the sequel as [Sch]) showed that at low temperature T, there are values $B_1(T) \leqslant B_2(T)$ such that when $B < B_1(T)$, $\mu_{\Lambda(1/h),-,Bh}$ converges weakly to μ_- and when $B > B_2(T)$, the limit is μ_+. This says that the negative boundary condition dominates in the limit when $B < B_1(T)$ whereas the small external field dominates when $B > B_2(T)$. The question, then, is whether for $T < T_c$ there exists a critical value $B_0 = B_0(T) = B_1(T) = B_2(T)$ such that $\mu_{\Lambda(1/h),-,Bh}$ converges to μ_- when $B < B_0$ and to μ_+ when $B > B_0$. In the case of $d = 2$, this question was completely solved by Schonmann and Shlosman (1996).

One of their methods is to extend the surface order large deviation results of Ioffe (1994, 1995) to a full large deviation principle. For higher dimensions, Greenwood and Sun (1997) ([GS] hereafter) proved the criticality of a particular value $B_0(T)$ for all $T < T_c$, but only in terms of the convergence of average spins rather than in terms of weak convergence. This paper extends the results presented there by showing that for low temperature and the same critical value B_0, $\mu_{\Lambda(1/h), -, Bh}$ converges weakly to μ_- when $B < B_0$ and to μ_+ when $B > B_0$.

Let us now recall the definition of B_0 from [GS]. It is known that the spontaneous magnetization $m_T^* := E_+[\sigma_0]$ $(= -E_-[\sigma_0])$ is zero when $T > T_c$ and positive when $T < T_c$. The average of σ in $\Lambda(1/h)$, called the average spin, is defined by

$$X_{\Lambda(1/h)}(\sigma) := (1/h)^{-d} \sum_{x \in \Lambda(1/h)} \sigma_x, \qquad \forall \sigma \in \Omega \tag{1.4}$$

It is also well known that $X_{\Lambda(1/h)}$ converges a.s. as $h \searrow 0$ to $-m_T^*$ (m_T^*) under μ_- (μ_+).

Define, for all $t \in R$,

$$\bar{\Phi}(t) = \limsup_{h \searrow 0} h^{d-1} \log E_{\Lambda(1/h), -, 0}[\exp(t(1/h)^{d-1} X_{\Lambda(1/h)})] \tag{1.5}$$

The function $\bar{\Phi}(t)$, called pressure function in large deviation theory, is convex and continuous on R by Hölder's inequality. Define

$$B_0 = B_0(T) := 2T \sup\{t: \bar{\Phi}(t) = -m_T^* t\} \tag{1.6}$$

For all $T < T_c$, B_0 is nonnegative and for small T, it is positive ([GS]). It is still an open problem whether $B_0 > 0$ for all $T < T_c$. It is believed so, though. At the end of this section we see that B_0 converges to $2d$ as T goes to zero. We know from [GS] that for all $T < T_c$, if $B < B_0$ then for all $\varepsilon > 0$,

$$\mu_{\Lambda(1/h), -, Bh}(|X_{\Lambda(1/h)} - (-m_T^*)| \geqslant \varepsilon) \to 0, \qquad \text{as} \quad h \searrow 0 \tag{1.7}$$

whereas if $B > B_0$ then the average spin, $X_{\Lambda(1/h)}$, no longer converges to $-m_T^*$ under $\mu_{\Lambda(1/h), -, Bh}$. Now we state our main results.

Theorem 1. For all $T < T_c$, (a) $0 \leqslant B_0 \leqslant 2d/m_T^*$, and (b) $\mu_{\Lambda(1/h), -, Bh}$ converges weakly to μ_- as $h \searrow 0$ if $B < B_0$.

Theorem 2. For small temperature T and all $B > B_0$, $\mu_{\Lambda(1/h_n), -, Bh_n}$ converges weakly to μ_+ along a subsequence $h_n \searrow 0$. Moreover, $\mu_{\Lambda(1/h), -, Bh}$ converges weakly to μ_+ as $h \searrow 0$ if $B > 2d/m_T^*$.

Theorems 1 and 2 together yield that B_0 is a critical value of the balance parameter B for low temperature. Note that in Theorem 2 the weak convergence of $\mu_{\Lambda(1/h),\,-,\,Bh}$ to μ_+ is proved only along a subsequence $h_n \searrow 0$ for $B \in (B_0, 2d/m_T^*]$. It is reasonable that this should hold as $h \searrow 0$, but our method does not prove this. A sufficient condition for this, seen in the proof of Lemma 3 below, is that the limit in (1.5) exists.

Theorem 2 of [Sch] says that $\mu_{\Lambda(1/h),\,-,\,Bh}$ converges weakly to μ_- if $B < B_1(T) = 2d(\beta'/\beta)$, and to μ_+ if $B > B_2(T) = 2d(1 + \delta(T))$, where $\beta' = \beta - \log b$ and b is some constant, and $\delta(T)$ is a positive valued function which vanishes as $T \searrow 0$. Obviously, $B_1(T) < B_0 < B_2(T)$. As $T \searrow 0$, $B_1(T)$, B_0 and $B_2(T)$ all converge to $2d$. Hence the lower and upper bounds $B_1(T)$ and $B_2(T)$, obtained in [Sch], are increasingly accurate as $T \searrow 0$. Our contribution is to show that, for small T, there is a threshold between these bounds.

2. THE PROOFS

To prove Theorem 1(b) we first show that any weak limit μ of the Gibbs measures $\mu_{\Lambda(1/h),\,-,\,Bh}$ has the same one dimensional marginal first moments as μ_-. Then we use the monotonicity $\mu_- \leqslant \mu$, to conclude that $\mu_- = \mu$.

Lemma 1. For all $T < T_c$, $B < B_0$ and $x \in Z^d$,

$$\lim_{h \searrow 0} E_{\Lambda(1/h),\,-,\,Bh}[\sigma_x] = -m_T^* \quad (= E_{\mu_-}[\sigma_x]) \tag{2.8}$$

Proof. We will use the result about average spin (1.7) to prove (2.8). Suppose that $B < B_0$. The FKG inequality implies that

$$-m_T^* = \lim_{h \searrow 0} E_{\Lambda(1/h),\,-,\,0}[\sigma_x] \leqslant \liminf_{h \searrow 0} E_{\Lambda(1/h),\,-,\,Bh}[\sigma_x] \tag{2.9}$$

for all $x \in Z^d$. So we need only prove that

$$\limsup_{h \searrow 0} E_{\Lambda(1/h),\,-,\,Bh}[\sigma_x] \leqslant -m_T^* \tag{2.10}$$

for all $x \in Z^d$. We first show (2.10) for $x = 0$ and all $B < B_0$ and then extend it to all $x \in Z^d$. Fix $B_1 \in (B, B_0)$ and let $B/B_1 = 1 - \varepsilon$. Note that, for each x in the box $\Lambda(\varepsilon/h)$, there is a box $\Lambda_x(1-\varepsilon)/h)$ inside $\Lambda(1/h)$ and centered at x. By the FKG inequality and the translation invariance of Gibbs measures,

$$
\begin{aligned}
&E_{\Lambda(1/h),\,-,\,B_1 h}\left[\sum_{x\in\Lambda(1/h)}\sigma_x\right]\\
&\quad\geqslant E_{\Lambda(1/h),\,-,\,B_1 h}\left[\sum_{x\in\Lambda(1/h)\setminus\Lambda(\varepsilon/h)}\sigma_x\right]+\sum_{x\in\Lambda(\varepsilon/h)}E_{\Lambda_x((1-\varepsilon)/h),\,-,\,B_1 h}[\sigma_x]\\
&\quad\geqslant E_{\Lambda(1/h),\,-,\,0}\left[\sum_{x\in\Lambda(1/h)\setminus\Lambda(\varepsilon/h)}\sigma_x\right]+|\Lambda(\varepsilon/h)|\cdot E_{\Lambda_x((1-\varepsilon)/h),\,-,\,B_1 h}[\sigma_0]\\
&\quad= E_{\Lambda(1/h),\,-,\,0}\left[\sum_{x\in\Lambda(1/h)}\sigma_x\right]-E_{\Lambda(1/h),\,-,\,0}\left[\sum_{x\in\Lambda(\varepsilon/h)}\sigma_x\right]\\
&\qquad+|\Lambda(\varepsilon/h)|\cdot E_{\Lambda((1-\varepsilon)/h),\,-,\,Bh/(1-\varepsilon)}[\sigma_0]\\
&\quad\geqslant E_{\Lambda(1/h),\,-,\,0}\left[\sum_{x\in\Lambda(1/h)}\sigma_x\right]-|\Lambda(\varepsilon/h)|\,(-m_T^*)\\
&\qquad+|\Lambda(\varepsilon/h)|\cdot E_{\Lambda((1-\varepsilon)/h),\,-,\,Bh/(1-\varepsilon)}[\sigma_0]
\end{aligned}
$$

In the last step we used that, by FKG, $E_{\Lambda(1/h),\,-,\,0}[\sigma_x]$ increases to $-m_T^*$ as $h\searrow 0$. Dividing the above inequalities by $|\Lambda(1/h)|$ and taking the upper limit, we get

$$\limsup_{h\searrow 0} E_{\Lambda(1/h),\,-,\,B_1 h}[X_{\Lambda(1/h)}]\geqslant -m_T^*+\varepsilon^d m_T^*+\varepsilon^d\limsup_{h\searrow 0}E_{\Lambda(1/h),\,-,\,Bh}[\sigma_0]$$

By (1.7), since $B_1<B_0$, the LHS equals $-m_T^*$. This proves (2.10) for $x=0$ and all $B<B_0$. Now let $x\in Z^d$ be arbitrary. Then $x\in\Lambda(1/h)$ for all small $h>0$. Define $h'>0$ such that $1/h'=2\,|x|+1/h$. Then $h'<h$, $h/h'\to 1$ as $h\searrow 0$ and the box $\Lambda_x(1/h')$ centered at x contains $\Lambda(1/h)$. Choose $B_1\in(B,B_0)$. Note that $Bh/h'\leqslant B_1$ for small $h>0$. By the FKG inequality,

$$
\begin{aligned}
E_{\Lambda(1/h),\,-,\,Bh}[\sigma_x]&\leqslant E_{\Lambda_x(1/h'),\,-,\,Bh}[\sigma_x]\\
&=E_{\Lambda(1/h'),\,-,\,B(h/h')\,h'}[\sigma_0]\\
&\leqslant E_{\Lambda(1/h'),\,-,\,B_1 h'}[\sigma_0]
\end{aligned}
$$

for small $h>0$. Combining this with (2.10) for $x=0$, we obtain (2.10) for $x\in Z^d$.

The following Lemma 2 is from Liggett (1985, Corollary 2.8, p. 75). For $\eta,\xi\in\Omega$, define $\eta\leqslant\xi$ if $\eta(x)\leqslant\xi(x)$ for all $x\in Z^d$. A function f on Ω is said to be increasing if $f(\eta)\leqslant f(\xi)$ whenever $\eta\leqslant\xi$. For any two probability measures μ_1 and μ_2, define $\mu_1\leqslant\mu_2$ if $E_{\mu_1}[f]\leqslant E_{\mu_2}[f]$ for all bounded local increasing functions f on Ω.

Lemma 2. Let μ_1 and μ_2 be two probability measures on Ω such that $\mu_1 \leqslant \mu_2$. Suppose they have the same one dimensional marginal first moments, that is $E_{\mu_1}[\sigma_x] = E_{\mu_2}[\sigma_x]$ for all $x \in Z^d$. Then $\mu_1 = \mu_2$.

Proof of Theorem 1. (a) It was proved in [GS] that $\bar{\Phi}(t) = -m_T^* t$ for $t \leqslant 0$. Hence $B_0 \geqslant 0$. Changing the negative boundary condition to positive and using the fact that $Z_{\Lambda(1/h),-,0} = Z_{\Lambda(1/h),+,0}$ and Jensen's inequality, one has

$$\begin{aligned}\log E_{\Lambda(1/h),-,0}&[\exp(t(1/h)^{d-1} X_{\Lambda(1/h)})]\\ &\geqslant \log[\exp(-2\beta d(1/h)^{d-1}) \cdot E_{\Lambda(1/h),+,0}[\exp(t(1/h)^{d-1} X_{\Lambda(1/h)})]\\ &\geqslant -2\beta d(1/h)^{d-1} + t(1/h)^{d-1} E_{\Lambda(1/h),+,0}[X_{\Lambda(1/h)}] \qquad (2.11)\end{aligned}$$

Hence $\bar{\Phi}(t) \geqslant -2\beta d + m_T^* t$ for all $t > 0$. By the definition of B_0, it cannot exceed $2T$ times the value of t such that $-m_T^* t = -2\beta d + m_T^* t$, that is, $B_0 \leqslant 2d/m_T^*$.

(b) The Gibbs measures $\mu_{\Lambda(1/h),-,Bh}$ are weakly relatively compact since Ω is compact. Let μ be any weak limit of $\mu_{\Lambda(1/h),-,Bh}$ along a subsequence. The FKG inequality implies that $\mu_- \leqslant \mu$. By Lemma 1, μ and μ_- have the same one dimensional marginal first moments. Lemma 2 implies that $\mu = \mu_-$, and we have weak convergence of $\mu_{\Lambda(1/h),-,Bh}$ to μ_- as $h \searrow 0$. ∎

Theorem 2 is an extension of Theorem 2 (ii) of [Sch] in which he proved the weak convergence of $\mu_{\Lambda(1/h),-,Bh}$ to μ_+ as $h \searrow 0$ for low temperature T and B exceeding some constant $B_2(T)$. We will use his strategy to prove Theorem 2.

The idea is to show that for $B > B_0$, with high probability under $\mu_{\Lambda(1/h),-,Bh}$ there will appear a large contour in $\Lambda(1/h)$, and then to show that as $h \searrow 0$, this large contour will eventually cover the whole space, and hence the Gibbs measures $\mu_{\Lambda(1/h),-,Bh}$ converge to μ_+ as $h \searrow 0$.

Contours are defined as usual ([Sch]). For any contour γ, let $\Theta(\gamma)$ be the set of sites inside of γ. To study the occurrence of large contours, we denote by $A_{h,\varepsilon}$ the set of configurations in $\Omega_{\Lambda(1/h),-}$ which have at least one contour surrounding a number of sites larger than the volume of $\Lambda(\varepsilon/h)$, i.e.,

$$A_{h,\varepsilon} = \{\sigma \in \Omega_{\Lambda(1/h),-} : \exists \text{ a contour } \gamma \text{ in } \sigma \text{ such that } |\Theta(\gamma)| > |\Lambda(\varepsilon/h)|\}$$

Note that $A_{h,\varepsilon}$ (with h, ε fixed) is increasing, in the sense that its indicator is an increasing function as defined earlier, because of the negative boundary condition. The first step is to show that the probability $\mu_{A(1/h),\,-,\,Bh}(A_{h,\varepsilon})$ converges to 1 as $h \searrow 0$ for $B > B_0$ and ε in a certain range.

Let b be the combinatorial constant defined by [Sch] in his expression (4). It appears in a bound for the number of families of contours satisfying certain constraints which we need not set out here. We will use inequality (44) of [Sch], which requires the quantity $\beta' := \beta - \log b$ to be positive. Later we will use that $\beta'/\beta \to 1$ as T goes to 0.

Lemma 3. Suppose $\beta > \log b$.

(a) For each $B \in (B_0, 2d/m_T^*]$, there exists a subsequence $h_n \searrow 0$ such that for all $\varepsilon < 2d(\beta'/\beta)/B$ there exists a $c > 0$ such that,

$$\mu_{A(1/h_n),\,-,\,Bh_n}(A_{h_n,\varepsilon}) \geqslant 1 - e^{-(1/h_n)^{d-1}(c+o(1))} \tag{2.12}$$

(b) If $B > 2d/m_T^*$, then for all $\varepsilon < m_T^* \beta'/\beta$ there exists a $c > 0$ such that

$$\mu_{A(1/h),\,-,\,Bh}(A_{h,\varepsilon}) \geqslant 1 - e^{-(1/h_n)^{d-1}(c+o(1))} \tag{2.13}$$

Remark 1. If $B \in (B_0, 2d/m_T^*]$, we have $2d(\beta'/\beta)/B \geqslant m_T^* \beta'/\beta$. Then the ε in both (a) and (b) can take values up to $m_T^* \beta'/\beta$, which depends only on the temperature T and converges to 1 when T goes to 0, since $\lim_{T \searrow 0} m_T^* = 1$.

Proof. The proof is a modification of that of Lemma 8 in [Sch]. Let $A = \varepsilon B$. For any subset E of Ω, define

$$Z_{A,\,-,\,s}(E) = \sum_{\sigma \in E \cap \Omega_{A,\,-}} \exp(-\beta H_{A,\,-,\,s}(\sigma)).$$

Let $\mathscr{R}_{\varepsilon/h}$ denote the complement of $A_{h,\varepsilon}$. Then

$$\begin{aligned}
\mu_{A(1/h),\,-,\,Bh}(A_{h,\varepsilon}^c) &= \mu_{A(1/h),\,-,\,Bh}(\mathscr{R}_{\varepsilon/h}) \\
&= \frac{Z_{A(1/h),\,-,\,Bh}(\mathscr{R}_{\varepsilon/h})}{Z_{A(1/h),\,-,\,0}(\mathscr{R}_{\varepsilon/h})} \frac{Z_{A(1/h),\,-,\,0}(\mathscr{R}_{\varepsilon/h})}{Z_{A(1/h),\,-,\,0}} \frac{Z_{A(1/h),\,-,\,0}}{Z_{A(1/h),\,-,\,Bh}} \\
&\leqslant \frac{Z_{A(1/h),\,-,\,Bh}(\mathscr{R}_{\varepsilon/h})}{Z_{A(1/h),\,-,\,0}(\mathscr{R}_{\varepsilon/h})} \frac{Z_{A(1/h),\,-,\,0}}{Z_{A(1/h),\,-,\,Bh}} \\
&= \frac{Z_{A(B/h'),\,-,\,h'}(\mathscr{R}_{A/h'})}{Z_{A(B/h'),\,-,\,0}(\mathscr{R}_{A/h'})} \frac{Z_{A(1/h),\,-,\,0}}{Z_{A(1/h),\,-,\,Bh}}, \qquad (h' = Bh).
\end{aligned} \tag{2.14}$$

Under our assumption $A = \varepsilon B < 2d\beta'/\beta$. From inequality (44) of [Sch], which holds for all $B > 0$,

$$\limsup_{h' \searrow 0} (h')^{d-1} \log \frac{Z_{A(B/h'),\,-,\,h'}(\mathscr{R}_{A/h'})}{Z_{A(B/h'),\,-,\,0}(\mathscr{R}_{A/h'})} \leqslant -(\beta/2)\, m_T^* B^d \tag{2.15}$$

for $\varepsilon < 2d(\beta'/\beta)/B$. Note that $h' = hB$. So

$$\limsup_{h' \searrow 0} (h)^{d-1} \log \frac{Z_{A(B/h'),\,-,\,h'}(\mathscr{R}_{A/h'})}{Z_{A(B/h'),\,-,\,0}(\mathscr{R}_{A/h'})} \leqslant -m_T^* \beta B/2 \tag{2.16}$$

for $\varepsilon < 2d(\beta'/\beta)/B$. On the other hand, it is not difficult to see that for $h > 0$,

$$\frac{Z_{A(1/h),\,-,\,Bh}}{Z_{A(1/h),\,-,\,0}} = E_{A(1/h),\,-,\,0}\left[\exp\left(\frac{\beta B}{2}(1/h)^{d-1} X_{A(1/h)}\right)\right] \tag{2.17}$$

Now choose a subsequence so that

$$\bar{\Phi}\left(\frac{\beta B}{2}\right) = \lim_{n \to \infty} h_n^{d-1} \log E_{A(1/h_n),\,-,\,0}\left[\exp\left(\frac{\beta B}{2}(1/h_n)^{d-1} X_{A(1/h_n)}\right)\right] \tag{2.18}$$

The definition of B_0 says that $\bar{\Phi}(\beta B/2) > -m_T^* \beta B/2$ for all $B > B_0$. From (2.14), (2.16) and (2.25) (note that the limit exists in (2.25)), we have

$$\lim_{h_n \searrow 0} h_n^{d-1} \log \mu_{A(1/h_n),\,-,\,Bh_n}(A_{h_n,\,\varepsilon}^c) \leqslant -m_T^* \beta B/2 - \bar{\Phi}(\beta B/2) < 0.$$

This proves (2.12).

To prove (b), let $B > 2d/m_T^*$. Using (2.17) and (2.11), we obtain that

$$h^{d-1} \log \frac{Z_{A(1/h),\,-,\,Bh}}{Z_{A(1/h),\,-,\,0}} \geqslant -2\beta d + \frac{\beta B}{2} E_{A(1/h),\,+,\,0}[X_{A(1/h)}].$$

By (2.14) and (2.16),

$$\lim_{h \searrow 0} h^{d-1} \log \mu_{A(1/h),\,-,\,Bh}(A_{h,\,\varepsilon}^c) \leqslant -m_T^* \beta B/2 + 2\beta d - m_T^* \beta B/2$$

$$= -m_T^* \beta (B - 2d/m_T^*) < 0 \tag{2.19}$$

for $\varepsilon < 2d(\beta'/\beta)/B$. This proves (2.13) for $\varepsilon < 2d(\beta'/\beta)/B$. Finally, if $\varepsilon_1 < m_T^* \beta'/\beta$, choose B_1 such that $B > B_1 > 2d/m_T^*$ and $\varepsilon_1 < 2d(\beta'/\beta)/B_1$. Note that $\mu_{A(1/h),\,-,\,Bh}(A_{h,\,\varepsilon_1}) \geqslant \mu_{A(1/h),\,-,\,B_1 h}(A_{h,\,\varepsilon_1})$ since $A_{h,\,\varepsilon_1}$ is an increasing set. Inequality (2.13) for ε_1 follows from (2.19) for $B = B_1$ and $\varepsilon = \varepsilon_1$. ∎

Proof of Theorem 2. The proof is much like the proof of Theorem 2 (ii) in [Sch]. Let $\mathscr{B}$ be the set of all $\sigma \in \Omega_{\Lambda(1/h), -}$ such that the infinite cluster of negative spins in σ intersects the box $\Lambda(1/(2h))$. The objective is to show that $\mu_{\Lambda(1/h), -, Bh}(\mathscr{B})$ converges to zero as $h \searrow 0$. Now let $\sigma \in \Omega$ and let $\mathscr{C}(\sigma)$ be the set of all sites in Z^d which belong to infinite clusters of negative spins in σ.

Suppose, for all $\alpha > 0$, we can find a subset $\mathscr{B}^\alpha$ of $\mathscr{B}$ such that for small $h > 0$,

$$\mathscr{B} \subset \mathscr{B}^\alpha \cup \left\{ \mathscr{G}_\alpha := \left\{ \sigma \in \Omega_{\Lambda(1/h), -} : |\mathscr{C}(\sigma) \cup \Lambda(1/h)| \geqslant \left(\frac{\alpha}{2^d - 1} \right)^d \left(\frac{1}{16h} \right)^d \right\} \right\} \tag{2.20}$$

Also suppose we can prove that for all $\alpha < (\beta/2 - \log b - \log 2)/(2\beta d)$ and $B > 0$,

$$\lim_{h \searrow 0} \mu_{\Lambda(1/h), -, Bh}(\mathscr{B}^\alpha) = 0. \tag{2.21}$$

Then we can finish the proof as follows. Let $\alpha_0 = 1/(8d)$, $l_0 = \alpha_0/(16(2^d - 1))$ and $\varepsilon_0 = (1 - l_0^d)^{1/d}$. Then there exists a $T_0 > 0$ such that for all $T < T_0$, we have $\alpha_0 < (\beta/2 - \log b - \log 2)/(2\beta d)$ and $\varepsilon_0 < m_T^* \beta'/\beta$. Now if $\sigma \in \mathscr{G}_{\alpha_0}$, the second set on the right hand side of (2.20), then for all contours γ in σ,

$$|\Theta(\gamma)| \leqslant \left(\frac{1}{h} \right)^d - \left(\frac{l_0}{h} \right)^d = \left(\frac{\varepsilon_0}{h} \right)^d = |\Lambda(\varepsilon_0/h)|,$$

which implies $\sigma \in A_{n, \varepsilon_0}^c$ by the definition of A_{n, ε_0}. Therefore we can apply Lemma 3 to obtain

$$\mu_{\Lambda(1/h), -, Bh}(\mathscr{G}_{\alpha_0}) \leqslant \mu_{\Lambda(1/h), -, Bh}(A_{h, \varepsilon_0}^c) \to 0 \tag{2.22}$$

as $h \searrow 0$ if $B > 2d/m_T^*$, and as $h \searrow 0$ along a subsequence if $B \in (B_0, 2d/m_T^*]$.

So from (2.20), (2.21) and (2.22) we conclude that for $T < T_0$,

$$\mu_{\Lambda(1/h), -, Bh}(\mathscr{B}) \to 0, \tag{2.23}$$

as $h \searrow 0$ if $B > 2d/m_T^*$, and as $h \searrow 0$ along the subsequence if $B \in (B_0, 2d/m_T^*]$.

Note that for each $\sigma \in \Omega_{\Lambda(1/h), -} \backslash \mathscr{B}$, there exists a contour in σ which surrounds the box $\Lambda(1/(2h))$ and such that the spins at the inner boundary of the contour are all positive. Therefore, the argument of [Sch], p. 19, conditioning on the contours and then using the Markov property of

Gibbs measures and the FKG inequality, gives that for any increasing local function f on Ω,

$$\liminf_{h \searrow 0} E_{\Lambda(1/h),\,-,\,Bh}[f] \geqslant E_{\mu_+}[f], \tag{2.24}$$

as $h \searrow 0$ if $B > 2d/m_T^*$, and as $h \searrow 0$ along the subsequence if $B \in (B_0, 2d/m_T^*]$. Theorem 2 now follows.

It now remains to define carefully a $\mathscr{B}^\alpha$ satisfying (2.20) and then prove (2.21). Basically speaking, for any $\sigma \in \mathscr{B}$, if the volume of $\mathscr{C}(\sigma) \cap \Lambda(1/h)$ has the order of the whole volume of $\Lambda(1/h)$, then we put it in $\mathscr{G}_\alpha$. Those $\sigma \in \mathscr{B}$ which do not have that many sites in their negative cluster in $\Lambda(1/h)$, are left in $\mathscr{B}^\alpha$. The delicate contour argument presented in [Sch] can then be used to show $\mathscr{B}^\alpha$ has small probability when h is small. Since our setting is a little different from that in [Sch], we point out the necessary changes. In [Sch], the Gibbs measures $\mu_{\Lambda(1/h),\,-,\,h}$ are defined in the box $\Lambda(B/h)$ with external field h, whereas our Gibbs measures in the box $\mathscr{B}(1/h)$ with external field Bh. This amounts to a change of variable in h. Let $V_i = \{x \in Z^d\colon \|x\|_\infty \leqslant i\}$. Define $I = \{i\colon \Lambda(3/(4h)) \subset V_i \subset \Lambda(1/h)\}$. A σ-chain is a set of sites $\{x_1, x_2, \ldots, x_n\} \subset Z^d$ at which σ has negative spin and $\|x_{i+1} - x_i\| = 1$, for all $1 \leqslant i \leqslant n-1$. Given $\sigma \in \Omega$, define, for each $i \in I$, $M_i(\sigma)$ as the set of sites $x \in V_i \cap \mathscr{C}(\sigma)$ which are connected to $\Lambda(1/(2h))$ by σ-chain entirely contained in V_{i-1} except, possibly, for its end point at x. Let $L_i(\sigma) = M_i(\sigma) \backslash V_{i-1}$. Define

$$\mathscr{B}^\alpha_{i,l} = \{\sigma \in \mathscr{B}\colon l = |L_{i+1}(\sigma)| \leqslant \alpha\, |M_i(\sigma)|^{(d-1)/d}\} \tag{2.25}$$

and $\mathscr{B}^\alpha = \bigcup_{i \in I,\, l \geqslant 1} \mathscr{B}^\alpha_{i,l}$.

For this $\mathscr{B}^\alpha$, (2.20) is obtained from Lemma 10 of [Sch] by replacing h there by $2dh$. To show (2.21), repeat the proof of Lemma 11 of [Sch], with the following changes. Replace $\Lambda(B/h)$, $\Lambda(3d/(2h))$, $\Lambda(2d/h)$, $\Lambda(d/h)$ there by $\Lambda(1/h)$, $\Lambda(3/(4h))$, $\Lambda(1/h)$, $\Lambda(1/(2h))$, respectively. Replace the external field h there by Bh. The coefficients of $1/h$ in the exponential components at the end of the proof will then be changed by a constant multiple, which does not change the result. The condition $B \geqslant 2d$ in Lemma 11 of [Sch] is not needed in our setting. It is imposed in [Sch] only to ensure that $\Lambda(2d/h) \subset \Lambda(B/h)$.

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